Math 275D Lecture 19 Notes

Daniel Raban

November 15, 2019

1 The Itô Integral as a Continuous Random Function

1.1 Constructing the integral as an a.s. limit of continuous function

Let $\mathcal{H} \subseteq L^2(\Omega \times [0,1])$ be the collection of **adapted** functions; that is, $f \in \mathcal{H}$ if $f(\omega, t) \in \mathcal{F}_t$ for all t. Last time we defined the Itô integral so that

$$I_T(f) = \lim_n I_T(f_n),$$

where f_n are simple functions with $f_n \xrightarrow{L^2} f$.

We wanted to construct a random function $F(\omega) \in C([0,T])$ such that $F_t(\omega) \stackrel{d}{=} I_t(f)$. Note that in $\int f(B_s)\, dB_s$, we had a Riemann sum:

$$\sum_k f(B(t_k))B(\Delta t_k)$$

We fixed ω first, cut the path B_t into small pieces, and took the limit of the sum. For $f \in \mathcal{H}$ we did not define $I_T(f)$ in this way. We defined $I_T(f)$ first, based on $f_n \to f$. So for each ω, we do not know what the path F_t looks like.

Theorem 1.1. *With probability 1, $t \mapsto F_t$ is continuous.*

Proof. Suppose

$$f_n = \sum_k a_k^{(n)} \mathbb{1}_{[t_k^{(n)}, t_{k+1}^{(n)}]}(t).$$

Then define

$$F_n(t) = \sum_{k \leq k_0} a_k^{(n)}(\omega)[B(t_{k+1}^{(n)}) - B(t_k^{(n)})] + a_{k_0+1}^{(n)}(\omega)B(t), \qquad k_0 = \max\{k : t_{k+1} \leq t\}.$$

This $F_n(t)$ is continuous a.s. We want to take the limit to get $F(t)$; that is, we want a limit point of $(F_n(t))$ in $C([0,1])$. So we want to find a Cauchy sequence in $C([0,T])$. We we want to look at

$$\max_t |F_n(t) - F_m(t)|$$

We defined F_n by $f(m)$, so if we denote $F_n = F(f_n)$, we see that $F_n - F_m = F(f_n - f_m)$. So we want to find

$$X_{m,n} := \max_t |F(f_n - f_m)(t)|$$

This difference, which we can call $F_{n,m}(t)$, is is a martingale with respect to t. Doob's inequality says

$$\mathbb{E}[X_{m,n}^2] \le C\,\mathbb{E}[F_{n,m}(T)^2]$$

So Chebyshev's inequality gives

$$\begin{aligned}\mathbb{P}(X_{m,n} \ge \varepsilon) &\le \frac{C}{\varepsilon^2}\,\mathbb{E}[F_{n,m}(T)^2] \\ &= \frac{C}{\varepsilon^2}\|I_T(f_n) - I_T(f_m)\|_{L^2}^2 \\ &= \frac{C}{\varepsilon^2}\|f_n - f_m\|_2^2.\end{aligned}$$

Pick a subsequence such that $\|f_{k_n} - f_{k_m}\|_2^2 \le 2^{-3n}$ for $k_m \ge k_n$. Then we get

$$\mathbb{P}(X_{n,m} \ge 2^{-n}) \le 2^{-n}.$$

By the Borel-Cantelli lemma, with probability 1, there is an $n_0(\omega)$ such that when $n \ge n_0$,

$$\|F_{k_n} - F_{k_{n+1}}\|_\infty \le 2^{-n}.$$

So F_{k_n} is Cauchy a.s. So f_{k_n} has a limit in $C([0,T])$. $\square$

1.2 Further considerations about this construction

1. For fixed t, does $F(t) \stackrel{d}{=} I_t(f)$? Yes, because $F_{k_n}(t) \stackrel{d}{=} I_t(f_{k_n})$ for each n.

2. $F(t)$ is also a martingale with respect to t: $F(s) = \mathbb{E}[F)t_{|}\mathcal{F}_s]$, so then $F_{k_n}(s) = \mathbb{E}[F_{k_n}(t) \mid F(s)]$. F is a mar

Based on how we defined this version of $F(t)$, it is unclear how much the path $F(\omega, t)$ depends on $B(\omega)$ for fixed ω. If we picked ω_0 so $B(\omega_0)$ is the 0 function, can we even say that $F(\omega_0)$ is the 0 function?